Hierarchical Model of Problem Solving: Social Interactions and Uncertainties

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I. Introduction

There is no exaggeration in saying that the question of how groups are formed to solve a problem is as old as the history of philosophical thought, and has been approached from either a normative, descriptive, or a positivist perspective. The current work argues that an attempt to try to understand and model the structure of constraints and incentives involved with the problems around which people come together seems to be more fruitful since it provides a theoretical and analytical framework to learn more about the formation of groups around them. An important step to understand these structures can be by the analysis of the notion that people interact with each other under around problems, and that these problems are organized in a hierarchical way. This reveals the cost and incentive structures associated with the questions at any level.

While solving problems in one-domain increases the chance the one will keep solving problems in the same domain (and decreases the chance that one will solve problems in a different domain). The costs, incentives, and rewards which are involved with the problem-solving process have also crucial role in determining people's decision of whether to stay in the same domain, or to solve a problem in different domains. Assume, for example, that a person chooses to study a degree in psychology. This increases the chance that the subsequent questions that will be asked by that person are in the domain of psychology. However, the incentives, costs of personal efforts of this person, more importantly his interaction with the others is of crucial importance in her decision to stay in the same domain, or to move the other fields. That is to say the structure of constraints and incentives involved with the problems leads to the formation of preferences to subsequent problems and this has an effect on the nature of group formation. As people become constrained by these constraints and incentives which kindles his/her interest in forming a group to solve a problem, they are also constrained in the type of groups they can form to solve problems. The model constructed here also aims to investigate the notions of group formation, with the assumption that problems are organized in hierarchies, in a way that determines the formation of groups.

Imagine an even plain field of problems. In this plain field, people may choose to solve problems whose solutions can lead to a large benefit compared to the cost and risk that are involved in solving these problems. Therefore, people assign different values to problems based on their potential individual reward (or cost) as well as the societal value (or cost) for pursuing these problems. These two factors (personal reward, societal reward) determine the effort, which is invested by the individual in perusing problems. Once a problem has been solved, the individual seeks to find the next problem in order to achieve a subsequent reward. However, unlike the case of an even plain field, previously choosing a certain problem leads to the formation of certain constrains on future choices (*figure 1*). This generally means that the amount of investment required in order to solve problems beyond these constrains has increased. Furthermore, our model assumes that individuals have imperfect information regarding the possible problems that occur beyond their local space.

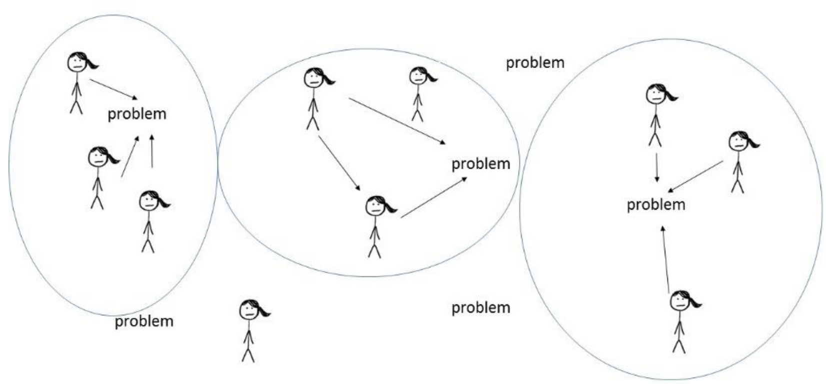

Figure 1 – Grouping constrains that are subsequent to a specific problem.

The notion that people seek to solve problems locally has implications on the ways the groups are formed around problems. As suggested by figure 2, the occurrence of a subsequent problem within a specific space determines the grouping around that space. Due to the existence of constrains, it is easier (higher effort-reward) for people within that constrain to pursue the problem than people outside of the constraint. This determines the subsequent grouping around the problem.

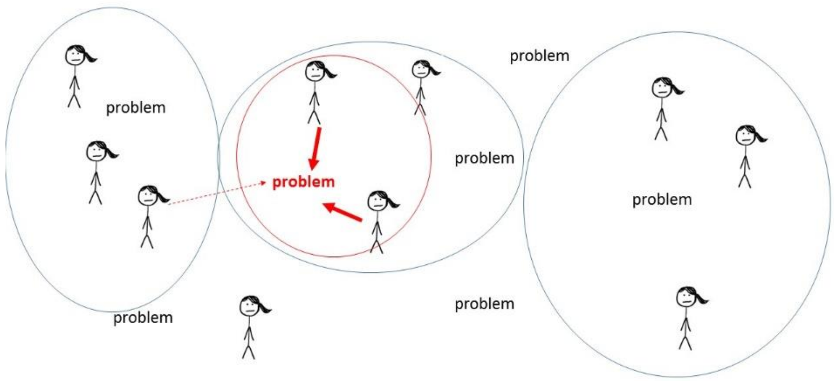

Figure 2 – Grouping constrains influence the formation of groups.

The following model aims at showing how people's social interactions determine preference structures for a certain problem.[1] Subsequently Shannon's information entropy and the maximum-entropy method[2] is employed to show that as long as a level of uncertainty exists, people try every available way to maximize their rewards by choosing and solving problems. In other words, the higher the reward of an action, the more frequently the agents do it in a non-linear, exponential setting.

II. Model

Consider a system of in which $N+1$ identical individual is seeking to solve a problem, and each of them controlling some action variable, a^i. Let the average social action of the rest of people at the macro-level be s such that the aggregate social action across the other agents is $\boldsymbol{s} = \frac{1}{N}\sum_{i=1}^{N} a^i$ which orchestrates the actions of the $N+1st$ individual. Thus, the scope and limits of rewards that a question promises is determined by the actions both she has control over and cannot command individually, i.e., her own action and the social action taken by the other agents at the

[1] For a discussion of this class of models see Chapter IV of Bowles's Microeconomics(2004), and see Duncan K. Foley's Social Interaction Models and Keynes's Macroeconomics (2015) for a beautiful presentation of formalization of social interactions.

[2] The definition of information entropy as a measure of information, choice, and uncertainty was introduced first by Shannon's classic 1948 paper, *A Mathematical Theory of Communication*, which marks the birth of modern information theory. It is worth noting that the maximum entropy principle was introduced by E. T. Jaynes's *Information Theory and Statistical Mechanics* (1957).

aggregate level. To keep the model as simple as possible, the decision problem to form a group to solve a problem can express itself through the following function:

$$\boldsymbol{r}(\boldsymbol{a},\boldsymbol{s}) = -\frac{1}{2}a^2 + \alpha a + \beta as \qquad (\mathbf{1})$$

where α is the private cost or benefit of the action taken individually and β is the manifestation of the effect of the synergy created by the interactions among the economic agents. This suggests that when $0 < \beta$, the social action at the aggregate level complements the one at the individual level, that there is a positive feedback from the combined result of the actions taken by the others to the individual action, and vice versa. Now, it is straightforward to show that the best individual response to solve a question is:

$$\boldsymbol{a}(\boldsymbol{s}) = \alpha + \beta \cdot s. \qquad (\mathbf{2})$$

and, since the identical agents choose the same action, the equilibrium condition requires $a = s$ so that when there is an internal equilibrium, it will be:

$$\boldsymbol{a}^* = \frac{\alpha}{1-\beta} \qquad (\mathbf{3})$$

which implies that when $1 < \beta$, an unstable equilibrium is obtained. However, when the individual action is bounded such that $a_{min} \leq a \leq a_{max}$, there is at least a stable equilibrium at $\boldsymbol{a}^* = a_{min}$ or $\boldsymbol{a}^* = a_{max}$ or both:

$$\boldsymbol{a}^* = a_{min} if \ \frac{\alpha}{1-2\beta} \leq a_{min} \qquad (\mathbf{4})$$

$$\boldsymbol{a}^* = a_{max} if \ \frac{\alpha}{1-2\beta} \geq a_{max} \qquad (\mathbf{5})$$

$$\boldsymbol{a}^* = \frac{\alpha}{1-2\beta} \ otherwise \qquad (\mathbf{6})$$

This means that a system of analyzing the decision to form a group to solve a problem in the light of rewards that the question promises and the incentives that emerges out of the social interactions will inevitably face with multiple equilibria The analysis of multiple equilibria points of the system is straightforward if individuals are viewed as decision makers whose decision is limited due to certain constrains. Such constrains can be viewed as communication channels with limited bandwidth, i.e., as capacity-limited agents in terms of their limited computational power and information processing capacities. To create such a model so, let's consider that individuals are choosing a strategy, a frequency distribution $\boldsymbol{f}(\cdot): A \rightarrow R$ over the action set to maximize the expected reward, where their objective function is expressed in terms of their expected reward $\boldsymbol{r}(\cdot): A \rightarrow R$, and all the possible constraints such as capability limitations, and economic constraints, etc., are embedded in the action set A. Then, the decision problem expresses itself in terms of maximizing over all the frequency distribution as follows:

$$\underset{(f(a|s)\geq 0 | \int f(a,s)da=1)}{\mathbf{Max}} \int f(a|s)r(a,s)da \mid a \in A. \qquad \mathbf{(7)}$$

Now, one appropriate way to introduce more realism into the model to remove the case of zero-uncertainty would be to set up the maximization problem with a constraint on the entropy of the frequency distribution:

$$\underset{(f(a|s)\geq 0 | \int f(a,s)da=1)}{\mathbf{Max}} \int f(a|s)r(a,s)da \mid a \in A \qquad \mathbf{(8)}$$

subject to

$$\boldsymbol{I(f)} = -\int f(a|s)\ln\big(f(a|s)\big)\,da \geq I_{min}. \qquad \mathbf{(9)}$$

where I_{min} is the minimum entropy. Then, when Lagrange's method is used to maximize entropy, the Lagrangian is:

$$\boldsymbol{L(f;\lambda)} = \int f(a|s)r(a,s)d\square + \lambda\left(-\int f(a|s)\ln\big(f(a|s)\big)\,da - I_{min}\right) \qquad \mathbf{(10)}$$

where the first order condition boils down to:

$$0 = r(a,s) - \lambda ln(f(a|s), \tag{11}$$

which means that $f(a|s)$ is proportional to $e^{\frac{r(a,s)}{\lambda}}$. The normalized solution to this problem for a given λ is the *Boltzmann-Gibbs distribution*:

$$\boldsymbol{f(a|s;\lambda)} = \frac{e^{\frac{r(a,s)}{\lambda}}}{\int e^{\frac{r(a,s)}{\lambda}} da}. \tag{12}$$

This implies that the decisions to form a group to solve a problem are made according to frequency distribution, which means that the higher the reward of action a, the more frequently the agents do it in a non-linear, exponential setting.

More importantly, the maximum-entropy method states that as the Lagrangian multiplier λ, which corresponds to 'temperature' in physics, get smaller, its results converge where all the frequency is put on the reward-maximizing question. An important implication of this result is that the agents perform every available action a with some positive frequency as long as $\lambda > 0$. At absolute zero entropy, the agents' actions are entirely concentrated on the reward maximizing choice. This means that at the limit point, all people concentrate at a single decision to form a group to solve the maximum reward question. This extreme case implies that zero-uncertainty, which can be made possible *if and only if* each individual is able to collect, carry, and process 'infinitely many information sets'. More specifically, this extreme implication is due to the fact that at the limit point**,** the probability of forming a group by all the individuals to solve the problem that gives the maximum reward goes to zero, i.e., $\boldsymbol{f(a|s;\lambda) \to 0}$.

III. Conclusion

To conclude, our model follows the idea that people organize around problems and that in order to understand the structure of organization one has to understand the structure of constraints and incentives involved with the problems around which people come. It has been argued that the problems are organized in a hierarchical way as a result of these structural constraints and incentives under uncertainties that determines the organization of groups. It is obvious that there are a number of ways to extent the model discussed here. First and foremost, the cost of solving a problem to society and some other forms of subsidies or incentives that kindles one's interest in solving a particular question should be explored. Information entropy and thermodynamic entropy is ready to provide a formal and fruitful way to analyze a model of the interactions, incentives, and cost structures that shapes people's decision to form groups to solve problems.